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A CANONICAL ARITHMETIC QUOTIENT FOR ACTIONS OF LATTICES IN SIMPLE GROUPS

BY

DAVID FISHER

Department of Mathematics, University of Chicago
5734 S. University Ave, Chicago, IL 60615, USA
e-mail: david@math.uchicago.edu

ABSTRACT

In this paper we produce an invariant for any ergodic, finite entropy action of a lattice in a simple Lie group on a finite measure space. The invariant is essentially an equivalence class of measurable quotients of a certain type. The quotients are essentially double coset spaces and are constructed from a Lie group, a compact subgroup of the Lie group, and a commensurability class of lattices in the Lie group.

1. Introduction

In his 1986 ICM address [Z2], Zimmer announced a program to classify all actions of lattices in simple groups by comparing them to certain arithmetically constructed actions. In this paper we give a natural ordering on the arithmetically constructed quotients of an arbitrary action and show that with respect to this ordering there is a unique maximal arithmetic quotient for any finite measure preserving, finite entropy, ergodic action. This quotient can be thought of as an invariant of the original action and can be described by a giving a Lie group, a compact subgroup and a commensurability class of lattices in the Lie group. This work is closely related to work of Lubotzky and Zimmer on actions of connected simple groups [LZ2].

One of the principal ingredients in our proof is Ratner theory. Ratner completely classified the invariant measures for actions of connected semisimple and unipotent groups on certain homogeneous spaces [R]. We use work of Witte and

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Shah that generalizes this classification to actions of lattices in simple groups [S, W].

In §6 we discuss related results from [F] where it is shown that for a continuous action on a manifold M, there are quotients associated to representations of the fundamental group of M. These provide lower bounds on the maximal quotient of this paper.

2. Preliminaries

We first define some basic terminology from ergodic theory. If a group G acts on a measure space (X,μ), we call a G-space (Y,ν) a **quotient of the G action** if there is a measure preserving G map $X \to Y$. If the G action on X is ergodic, then the action on Y will be ergodic and the map will be essentially surjective. By a **virtual quotient** of X we mean a quotient of a finite extension of the G action on X.

Example 2.1: Suppose G acts on X and on $X' \to X$, a finite cover. Then if Y is a quotient for the G action on X', it is a virtual quotient for the G action on X. This includes the case of disconnected covers. In particular, if $G_0 < G$ is a subgroup of finite index, then $X' = (X \times G)/G_0$ is a finite cover for X. This implies that any quotient for the G_0 action on X is a virtual quotient for the G action.

We now define an important class of actions for lattices $\Gamma < G$ where G is a simple Lie group.

Definition 2.2: An action of Γ is **arithmetic** if it is constructed in one of the following four ways:

1) Let $G \to H_{\mathbb{R}}$ be a group homomorphism, where H is a $\mathbb{Q}$ algebraic group, Λ an arithmetic lattice in H and $B < H_{\mathbb{R}}$ a compact group commuting with the image of G. Then any lattice $\Gamma < G$, acts on $B\backslash H_{\mathbb{R}}/\Lambda$.

2) Let N be a connected, simply connected nilpotent group, $\Delta < N$ a lattice. Suppose G acts on N by automorphisms so that Γ normalizes Δ. Then Γ acts on the nilmanifold N/Δ, and also on $D\backslash N/\Delta$, where $D < \mathrm{Aut}(N)$ is any compact subgroup commuting with Γ.

3) Let $\Gamma \to K$ be a group homomorphism with dense image where K is compact. Let $C < K$ be a subgroup. Then Γ acts on K/C.

4) Let Γ act diagonally on $Y = Y_1 \times Y_2 \times Y_3$ where Y_1 is as in **1**) above, Y_2 is as in **2**) above and Y_3 is as in **3**) above.

Remark: (i) The space Y in **4)** can be written as

$$(B\times D)\backslash(H\times N\times K)/(\Lambda\times\Delta\times C).$$

This is not quite a double coset space as D is not a subgroup of N. From the point of view of group actions it will usually be more convenient to view this space as described in **4)**.

(ii) For some groups G with $\mathbb{R}$-rank$(G) = 1$, there are algebraically defined Γ actions that are not arithmetic in this sense, since there are non-arithmetic lattices in some rank one groups.

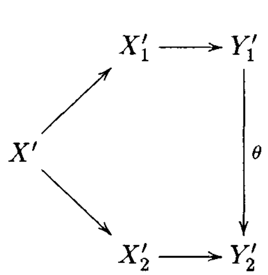
A **virtual arithmetic quotient** is a virtual quotient that is an arithmetic action. An action of a group is said to have **finite entropy** if the entropy of any element of the group is finite. Given two arithmetic actions of a lattice Γ in a simple group G, say $Y_i = (B\backslash H/\Lambda_i)\times(D\backslash N/\Delta_i)\times(K/C)$, for $i = 1,2$, where the action is given by the same representation of Γ, we call the actions **commensurable** if Λ_1 is commensurable to Λ_2 and Δ_1 is commensurable to Δ_2. Commensurable actions clearly admit a common finite extension.

We now define the ordering on virtual arithmetic quotients.

Definition 2.3: If $X_i\to Y_i = (B_i\backslash H_i/\Lambda_i)\times(D_i\backslash N_i/\Delta_i)\times(K_i/C_i)$ are virtual arithmetic quotients of X, we say that $Y_1\succ Y_2$ if by passing to commensurable virtual arithmetic quotients $X_i'\to Y_i'$ we can find
(i) a common extension X' of the X_i' and
(ii) three $\mathbb{Q}$ surjections:

$$\begin{array}{lcl} \theta_1 : H_1\to H_2 & & \theta_1(B_1) < B_2 \\ & & \theta_1(\Lambda_1') < \Lambda_2' \\ \theta_2 : N_1\to N_2 & \text{such that} & \theta_2(D_1) < D_2 \\ & & \theta_2(\Delta_1') < \Delta_2' \\ \theta_3 : K_1\to K_2 & & \theta_3(C_1) < C_2 \end{array}$$

so that



commutes, where $\theta = (\theta_1,\theta_2,\theta_3)$.

The main result of this paper proves the existence of a maximal virtual arithmetic quotient for any ergodic action of a lattice in simple group (with no rank assumption) with finite invariant measure and finite entropy. Actually, the maximal quotient will be an equivalence class of virtual arithmetic quotients, that differ in arithmetic construction, but that have measurably isomorphic finite extensions.

We prove this result by dealing with each of the three different types of arithmetic quotients individually and then showing that the product of the maximal object of each type is indeed the maximal arithmetic quotient desired. From now on we refer to arithmetic actions and quotients of type **1)** in Definition 2.2 as G quotients, those of type **2)** as nilpotent quotients and those of type **3)** as isometric quotients. We will also refer to virtual G quotients, virtual nilpotent quotients and virtual isometric quotients.

Theorem 2.16 of [LZ2] shows that an ergodic action of a connected simple group G with finite entropy has a maximal virtual G quotient unique up to commensurability and $\mathbb{R}$ conjugacy (see Definition 5.1 below). In fact, the proof of [LZ2] also shows the existence of a unique maximal G quotient for a Γ action provided we use the results of Shah and Witte in place of Ratner's Theorem. It is interesting to note that their proof of the existence of a maximal quotient goes through without this modification. We only need the results of Shah and Witte to prove uniqueness, since here we study the push-forward of a Γ invariant measure from the original space X to a homogeneous space, and it is only from the classification of invariant measures that we can see that this Γ invariant measure is in fact G invariant.

To complete the proof of our main result, we now need only show the existence of maximal nilpotent and maximal isometric quotients, and then show that the product of the maximal object of each type is the maximal virtual arithmetic quotient.

3. Nilpotent quotients

In this section, we will show the existence of a maximal virtual nilpotent quotient. Throughout we will refer to an action of Γ on $D\backslash N/\Delta$ as described in **2)** of Definition 2.2 as a **nilpotent action**. The first step in the proof is to show that the finite entropy condition bounds the dimension of the quotients. The uniqueness of the maximal quotient will then follow from Ratner theory which will be used to show that any pair of virtual nilpotent quotients is dominated by a common virtual nilpotent quotient. The theorem we need here is proven in

[W]:

THEOREM 3.1: *Let Γ be a lattice in a connected semisimple Lie group G with no compact factors and let N be a connected, simply connected nilpotent group and $\Lambda < N$ a lattice. Suppose G acts on N by automorphisms such that Γ normalizes Λ and so acts on N/Λ. Then any ergodic Γ invariant probability measure μ on N/Λ is homogeneous for a subgroup of $\Gamma \ltimes N$, i.e., is the smooth measure on a closed orbit of a closed subgroup of $\Gamma \ltimes N$.*

We need the following standard fact about the entropy of lattice actions on nilmanifolds.

LEMMA 3.2: *Suppose Γ acts on N via automorphisms. Let $\Lambda < A < G$ be the maximal $\mathbb{R}$-split subgroup of Γ. Then for $\gamma \in A$, we have the following formula for the entropy of the action:*

$$h_{N/\Delta}(\gamma) = h_{K\backslash N/\Delta}(\gamma) = \Sigma \log \omega(\gamma)$$

where ω is a weight of the G representation on $\mathfrak{n}$ with respect to A, and the sum is over $\omega(\gamma) > 1$.

The finite entropy condition on the Γ action on X controls the possible quotients for the actions.

COROLLARY 3.3: *If $\Gamma < G$ is a lattice in a simple group and Γ acts ergodically on X with finite invariant measure and finite entropy, then the set of entropy functions $\{h(\gamma) | \gamma \in A\}$ for virtual nilpotent quotients of X is finite.*

Proof: This follows from the proceeding lemma and standard facts about representation theory, once we note the following: if Y is a quotient of X, then $h_X \geq h_Y$ and equality occurs when the quotient is a finite extension. ∎

To facilitate our use of Theorem 3.1 we need to work with ergodic actions on N/Λ rather than $K\backslash N/\Lambda$. This motivates the following definition and lemma.

Definition 3.4: Let Γ acting on $K\backslash N/\Delta$ be a nilpotent action. We say the action is of **reduced form** if the Γ action on N/Δ is ergodic.

LEMMA 3.5: *Every nilpotent action of Γ has a finite extension which is a nilpotent action of reduced form.*

Proof: Consider the extension $N/\Delta \to K\backslash N/\Delta$. Since the action on the base is volume preserving and ergodic, we can pick an ergodic component of the measure on N/Δ that projects to the measure on the base. By Theorem 3.1, this is supported on an N' orbit for some subgroup $N' < N$, say $N'/N' \cap l\Delta l^{-1}$ for some N' which is normalized by Γ and some $l \in \mathrm{Aut}(N)$. Since the projection of this N' orbit to $K\backslash N/\Delta$ is of full measure, it follows that there is $y \in N$ such that $KN'y\Delta$ is of full dimension in N, and therefore so is KN. Since K is compact, $KN' = N$. Therefore $(K \cap \mathrm{Aut}(N'))\backslash N'/(N' \cap l\Delta l^{-1}) \to K\backslash N/\Delta$ is a surjective, measure preserving map of manifolds of the same dimension and finite volume. So it is a finite extension. ∎

We now prove the existence of maximal, though not necessarily unique, nilpotent quotients.

LEMMA 3.6: *There is a virtual nilpotent quotient in reduced form that is maximal.*

Proof: By Corollary 3.3, we can choose a virtual nilpotent quotient in reduced form, $C'\backslash N/\Delta$, with maximal entropy function among all such quotients. Now consider groups $C < C'$ such that $C\backslash N/\Delta$ is a virtual arithmetic quotient. By the descending chain condition on compact subgroups of C' there is a minimal such C.

Now we consider all larger virtual arithmetic quotients of reduced form $K\backslash M/\Lambda \succ C\backslash N/\Delta$. To prove the lemma it suffices to see that possible values of $\dim(M)$ are bounded. Let $D = \ker(M \to N)$. Then D is a $\mathbb{Q}$ group. Lemma 3.2 above shows that G and Γ centralize D, since $C\backslash N/\Delta$ already has maximal entropy.

Now let $P = G \ltimes M$. Then $Z_P(D)$ is a normal subgroup of P which contains G. Looking at the action of G on $G \ltimes M/\Gamma \ltimes \Lambda$ we see by Lemma 2.7 of [LZ2] that since $G < Z_P(D)$, then $Z_P(D) = P$ and so D is central in P and therefore in M. Since $\mathfrak{p} = \mathrm{Lie}(P)$ is defined over $\mathbb{Q}$, we can choose a $\mathbb{Q}$ subspace $\mathfrak{d}' \subset \mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{d} \bigoplus \mathfrak{d}'$. Let $B : \mathfrak{d}' \times \mathfrak{d}' \to \mathfrak{d}$ be given by $B(x,y) = \mathrm{proj}_{\mathfrak{d}}([x,y])$. Since $\mathfrak{d} \subset \mathfrak{z}(\mathfrak{p})$, the subspace $I = \mathfrak{d}' \bigoplus B(\mathfrak{d}' \times \mathfrak{d}')$ is an ideal in $\mathfrak{p}$ and I is defined over $\mathbb{Q}$.

Now $\dim(\mathfrak{d}') \leq \dim(G \ltimes N)$, so if $\dim(M) > \dim(N) + \dim(G \ltimes N)^2$, I is a proper $\mathbb{Q}$ ideal. Thus $J = [I, I]$ is a proper $\mathbb{Q}$ ideal that is also an algebraic Lie subalgebra. Since $\mathfrak{d}$ is unipotent, we can choose $\mathfrak{d}'$ such that some conjugate of $\mathfrak{g}$ is contained in $\mathfrak{d}'$, hence in I, and since $\mathfrak{g}$ is semisimple, in J. It follows from Lemma 2.7 of [LZ2] that some conjugate of G does not act ergodically on $G \ltimes M/\Gamma \ltimes \Lambda$. Since this action is conjugate to the action induced from the Γ action on M/Λ, we see that Γ doesn't act ergodically on M/Λ. This contradicts our assumption that $K\backslash M/\Lambda$ was in reduced form. Therefore

$$\dim(M)\leq\dim(N)+\dim(G\ltimes N)^2. \qquad \blacksquare$$

Two measurably isomorphic nilpotent actions can have different $\mathbb{Q}$ structures. Since we are working in the measurable category and would like to view these as the same action, we need the following definition.

Definition 3.7: Let $C\backslash N/\Delta$ be a nilpotent arithmetic Γ space defined via a map $\rho_1:\Gamma\to Z_{\mathrm{Aut}(N/\Delta)}(C)$ and $h\in\mathrm{Aut}(N/\Delta)$. Then Γ also acts on $hzCz^{-1}h^{-1}\backslash N/\Delta$ via $\rho_2:\Gamma\to Z_{\mathrm{Aut}(N/\Delta)}(hzCz^{-1}h^{-1})$ where $\rho_2(\gamma)=h\rho(\gamma)h^{-1}$. We call two such nilpotent arithmetic actions $\mathbb{R}$ **conjugate**.

The following theorem shows the existence of a maximal virtual nilpotent quotient that is unique up to commensurability and $\mathbb{R}$ conjugacy.

THEOREM 3.8: *Let $\Gamma<G$ be a lattice in a simple Lie group. Let X be an ergodic Γ space with finite invariant measure and finite entropy. Then*
(1) *There is a maximal virtual nilpotent quotient of the action in reduced form, say $N(X)$.*
(2) *$N(X)$ is unique up to $\mathbb{R}$ conjugacy.*
(3) *If Y is any virtual nilpotent quotient of X in reduced form, then $Y\prec Z$ where Z is an $\mathbb{R}$ conjugate of $N(X)$.*
(4) *Any virtual nilpotent quotient of X is the quotient of some finite ergodic extension of $N(X)$.*

Proof: Let $C\backslash N/\Delta$ be a maximal quotient from Lemma 3.6. Let $K\backslash M/\Lambda$ be any other virtual nilpotent quotient in reduced form. Passing to commensurable actions, there is a finite extension X' of X and measure preserving Γ maps ψ: $X'\to C\backslash N/\Delta$ and Ψ: $X'\to K\backslash M/\Lambda$. Let μ be the relevant measure on X'; then $\nu=(\psi,\Psi)_*\mu$ is a finite Γ invariant ergodic measure for the diagonal action on $(C\backslash N/\Delta)\times(K\backslash M/\Lambda)=(C\times K)\backslash(N\times M)/(\Delta\times\Lambda)$ that projects to the standard measure on each factor. Since both quotients are in reduced form, we can lift ν to an ergodic invariant measure ν' on $(N\times M)/(\Delta\times\Lambda)$ that projects to the standard measure on both factors and projects to ν. By Theorem 3.1, this measure is supported on the orbit of a Lie group $J<N\times M$ that is normalized by Γ. It follows that J projects surjectively to M and N. Thus the J orbit supporting ν' is J/Ξ where $\Xi=\Delta\times l\Lambda l^{-1}$ where $l\in\mathrm{Aut}(M)$. Give M the $\mathbb{Q}$ structure obtained by conjugating the given one by l. Then J is a $\mathbb{Q}$ subgroup and Ξ is

arithmetic and the surjection $J \to N$ is a $\mathbb{Q}$ surjection. Furthermore, the projection of J/Ξ to $(C\backslash N/\Delta)\times(K\backslash M/\Lambda)$ is $(\mathrm{Aut}(J)\cap(C\times K))\backslash J/\Xi)$. Thus letting $D = \mathrm{Aut}(J)\cap(C\times K)$, we have that $D\backslash J/\Xi$ is a nilpotent virtual arithmetic quotient of X in reduced form and that $D\backslash J/\Xi \succ C\backslash N/\Delta$ and so is commensurable to $C\backslash N/\Delta$. Via a conjugate $\mathbb{Q}$ structure on J we see that $D\backslash J/\Xi \succ K\backslash M/\Lambda$. The conjugate $\mathbb{Q}$ structure corresponds to an $\mathbb{R}$ conjugate action, and this suffices to prove the theorem. ∎

4. Isometric quotients

Since all isometric actions have zero entropy, we need a different set of techniques to find a maximal virtual isometric quotient. We depend on elementary facts about isometric actions and the following fact about representations of lattices, originally due to Margulis, but only stated explicitly in [Z3].

LEMMA 4.1: *There are only finitely many dense embeddings of Γ in compact Lie groups.*

Proof: By the argument of theorem 3.8 of [Z3], any such embedding is Galois conjugate to the structure of Γ as an arithmetic subgroup of a compact extension of G. This leaves us with only finitely many possibilities (up to conjugacy). ∎

COROLLARY 4.2: *There exist maximal isometric quotients.*

We now prove that maximal isometric quotients exist.

THEOREM 4.3: *Let $\Gamma < G$ be a lattice in a simple Lie group. Let X be an ergodic Γ space with finite invariant measure and finite entropy. Then:*
(1) There is a maximal virtual isometric quotient of the action, say $K(X)$.
(2) $K(X)$ is unique up to conjugacy.
(3) If Y is any virtual isometric quotient of X, then $Y \prec Z$ where Z is a conjugate of $K(X)$.
(4) Any virtual isometric quotient of X is the quotient of some finite ergodic extension of $K(X)$.

Proof: Let K/C be a maximal quotient which exists by Corollary 4.2. Let D/E be any other isometric virtual arithmetic quotient in reduced form. Passing to commensurable actions, there is a finite extension X' of X and measure preserving Γ maps $\psi\colon X' \to K/C$ and $\Psi\colon X' \to D/E$. Let μ be the relevant measure on X'; then $\nu = (\psi, \Psi)_*\mu$ is a finite Γ invariant ergodic measure for the diagonal action on $(K/C)\times(D/E)$ that projects to the standard measure on each factor.

Let $B = \overline{(\Gamma, \Gamma)} < K \times D$. Since ν is ergodic and Γ invariant, it will be supported on a single B orbit. This follows since the space of orbits of a compact group action is separated. Therefore ν is Haar measure on some B orbit, i.e., Haar measure on B/A for $A = B \cap (kCk^{-1} \times dEd^{-1})$ where $k \in K$ and $d \in D$. From the structure of ν, we see that we have maps ρ: $B \to K$ and ρ': $B \to D$ such that $\rho(A) < kCk^{-1}$ and $\rho'(A) < dEd^{-1}$. Conjugating both actions, we see that $B/A \succ K/C$ and $B/A \succ D/E$. Since K/C is maximal, it follows that $B/A = K/C$ and $K/C \succ D/E$. The rest of the theorem now follows. ∎

5. Maximal quotients

We now state the main theorem of this section, which shows that we get a maximal virtual arithmetic quotient simply by taking the diagonal action on the product of the three maximal quotients above. We will need to define $\mathbb{R}$ conjugacy for a general arithmetic quotient. First we give the definition for G quotients from [LZ2].

Definition 5.1: Let $K \backslash H/\Lambda$ be an arithmetic G-space defined via a homomorphism π_1: $G \to Z_H(K)$. Let $z \in Z_H(\pi_1(G))$ and $h \in H$. Then G also acts on $hzKz^{-1}h^{-1} \backslash H/\Lambda$ via π_2: $G \to Z_H(hzKz^{-1}h^{-1})$, where $\pi_2(g) = h\pi_1(g)h^{-1}$. We call two such arithmetic actions $\mathbb{R}$ **conjugate**.

The definition of $\mathbb{R}$ conjugacy for general arithmetic actions is now simple.

Definition 5.2: Two arithmetic actions are $\mathbb{R}$ **conjugate** if and only if the G actions and the nilpotent actions are $\mathbb{R}$ conjugate and the isometric actions are conjugate.

In this section we will need a more general statement about invariant measures than Theorem 3.1.

THEOREM 5.3: *Suppose Γ is a lattice in a semi-simple group G, $G < H$ a Lie group and Λ is a lattice in H. Any finite ergodic measure μ for the Γ action on H/Λ is homogeneous. I.e., there is a closed subgroup D, with $\Gamma < D < H$ such that μ is Haar measure on a closed D orbit.*

This theorem is stated in [S] as corollary 1.4 under weaker assumptions on G. The proof is attributed to D. Witte.

We can now state the main result of this section.

THEOREM 5.4: *Let $\Gamma < G$ be a lattice in a simple Lie group. Let X be an ergodic Γ space with finite invariant measure and finite entropy. Then:*

(1) *There is a maximal virtual arithmetic quotient of the action in reduced form, say* $A(X)$.
(2) $A(X)$ *is unique up to* $\mathbb{R}$ *conjugacy.*
(3) *If* Y *is any virtual arithmetic quotient of* X *in reduced form, then* $Y \prec Z$ *where* Z *is an* $\mathbb{R}$ *conjugate of* $A(X)$.
(4) *Any virtual arithmetic quotient of* X *is the quotient of some finite ergodic extension of* $A(X)$.

Proof: Let G(X) be the maximal G quotient from [LZ2], N(X) be the maximal nilpotent quotient, and K(X) the maximal isometric quotient. We claim that the diagonal action on $A(X) = G(X) \times N(X) \times K(X)$ is the maximal virtual arithmetic quotient of the theorem. To see this, it suffices to see that Haar measure on $A(X)$ is ergodic. It will then follow that $A(X)$ with Haar measure is a virtual quotient of X. We have maps θ_1: $X \to G(X)$, θ_2: $X \to K(X)$ and θ_3: $X \to N(X)$. Let ν be the push-forward under $(\theta_1, \theta_2, \theta_3)$ of the measure on X. This is a Γ invariant measure on $A(X)$ which projects to Haar measure on each factor. If Haar measure is ergodic, then any ergodic component of ν must equal the Haar measure and therefore ν must as well. The rest of the theorem then follows from Theorem 4.3, Theorem 2.16 and Theorem 2.16 of [LZ2]. So we are reduced to showing

LEMMA 3.5: *The diagonal action of* Γ *on* $G(X) \times N(X) \times K(X)$ *is ergodic.*

Proof: We have assumed that $G(X) = B \backslash H/\Lambda$ and $N(X) = D \backslash N/\Delta$ are in reduced form and that the action on $K(X) = K/C$ is given by a dense embedding of Γ in K. It therefore suffices to show that the Γ action on $(H/\Lambda) \times (N/\Delta) \times K$ is ergodic with respect to Haar measure. Let ν' be the push-forward of the measure on X to $G(X) \times N(X) \times K(X)$; this projects to Haar measure on each factor. This is an ergodic measure, and we can lift it to an ergodic measure ν on $H/\Lambda \times N/\Delta \times K = (H \times N \times K)/(\Lambda \times \Delta)$ that projects to ν' and that projects to Haar measure on each factor. Inducing to a G action and applying Theorem 5.3 we see that this measure is supported on a closed orbit of a closed subgroup $J < H \times N \times K$. We will show that $J = H \times N \times K$. By the general structure theory of Lie groups and our use of Theorem 3.5, we see that $J = L \times M \times C$ where $\Gamma < G < L$ and M is nilpotent with $G < \mathrm{Aut}(M)$ and C is compact, with a dense embedding $\Gamma < C$. The orbit supporting ν can be viewed as $(L \times M \times C)/(\Lambda' \times \Delta')$ where Δ' and Λ' are lattices in M and L, respectively. This space can be viewed as $(L/\Lambda') \times (M/\Delta') \times C$ and we will now examine all possible projections of each of these factors onto each factor of $(H/\Lambda) \times (N/\Delta) \times K$ to deduce our result. Since these projections are measurable and Γ equivariant, we can use the results of [W]. There Witte shows that any measurable quotient of such an action is actually affine. This forces L/Λ' to map trivially to N/Δ and K, and M/Δ' to map trivially to H/Λ and K. Also, since Γ is dense in C, any quotient of C has the form C/D for D a closed subgroup. This forces the projection of C into N/Δ and H/Λ to be trivial. Since neither of the other factors can map onto K, the map $C \to K$ must be a finite cover. Also, the maps $L/\Lambda' \to H/\Lambda$ and $M/\Delta' \to N/\Delta$ must be affine. Therefore, since $L/\Lambda' \times M/\Delta' \times C$ projects surjectively onto each of H/Λ, N/Δ and K, we have $H = L$, $M = N$, and $K = C$. Therefore we have exhibited Haar measure on $H/\Lambda \times N/\Delta \times K$ as an ergodic measure and the theorem follows. ∎ ∎ Theorem 5.4

6. Open questions and related results

In this section we briefly describe some open questions and related results. Once we have constructed a canonical virtual arithmetic quotient it is natural to ask what its relation will be to the original action. A very special form of this question is to ask when the canonical virtual arithmetic quotient will be non-trivial.

If $G = \mathrm{SO}(1,n)$ then there are lattices $\Gamma < G$ with homomorphisms $\rho\colon \Gamma \to \mathbb{Z}$. This implies that any finite measure preserving, finite entropy action of $\mathbb{Z}$ gives rise to a nontrivial Γ action. Elementary considerations of entropy and spectrum immediately imply that for most of these actions, the maximal quotient we construct must be trivial. This leads to the question of when the quotient will be non-trivial.

The situation is very different if we assume $\mathbb{R}$-rank$(G) \geq 2$. Then for any $\Gamma < G$, every known example of a finite measure preserving ergodic action of Γ on a manifold X is measurably isomorphic to a representative of its maximal quotient $A(X)$. Even for the exotic action of Katok–Lewis and Benveniste [KL,B], the action is measurably isomorphic to a finite union of ergodic arithmetic actions. Currently, however, there is no proof that $A(X)$ need be non-trivial for G of higher rank.

In [F], we use dynamical and topological hypothesis to prove that if Γ is a lattice in a higher rank simple group acting on a compact manifold M, then certain representations of the fundamental group $\pi_1(M)$ give rise to non-trivial quotients of the action on M.

We now briefly outline these results. The hypothesis we need are *engaging* conditions, which allow us to assume that lifts of the action to finite covers are as ergodic as the original action. Various engaging conditions exist for connected groups; see, e.g., [G, Z4]. In [F], we give the first general definition of engaging for discrete groups.

Definition 6.1 ([F]): Let Γ act on a manifold M preserving a measure. The action is **engaging** if for any choice of:

- finite index subgroup $\Gamma' < \Gamma$,
- finite cover M' of M,
- lift of the Γ' action to M',

any Γ' invariant measurable function on M' is a lift of a measurable Γ invariant function on M.

Given a discrete group Γ acting on a manifold M, let Λ be the group of lifts of the Γ action to the universal cover of M. Then Λ can be described by the exact sequence:

$$1 \to \pi_1(M) \to \Lambda \to \Gamma \to 1.$$

In Theorem 6.2 below, assume $\Gamma < G$ is a lattice with $\mathbb{R}$-rank$(G) \geq 2$, and $\pi_1(G)$ is finite.

THEOREM 6.2 ([F]): *Suppose Γ acts continuously on a compact manifold M, preserving finite measure and engaging. Further assume there is an infinite image linear representation $\sigma\colon \Lambda \to \mathrm{GL}_n(\mathbb{R})$. Then there exist a $\mathbb{Q}$ algebraic group J and a finite set of primes S such that $J_{\mathbb{Z}} < \sigma(\pi_1(M)) < J_{\mathbb{Z}_S}$. Furthermore, there is a finite index subgroup $\Gamma_0 < \Gamma$ and a finite cover M' of M such that there is a measurable Γ_0 equivariant map $\phi\colon M' \to Y$, where Y is one of:*

(1) *$C \backslash J_{\mathbb{R}}/J_{\mathbb{Z}}$. Here the Γ action is given by the existence of a perfect $\mathbb{Q}$ algebraic group $L = G \ltimes J$ so $\Gamma < G < L$ acts on $J/J_{\mathbb{Z}}$ and $C < Z_L(\Gamma)$ is compact.*

(2) *$C \backslash J_{\mathbb{R}}/J_{\mathbb{Z}}$. Here J is a perfect $\mathbb{Q}$ algebraic group and the Γ action is given by $\Gamma < G < J$ and $C < Z_J(G)$ is compact.*

(3) *Or $Y = Y_1 \times Y_2$, $J = J_1 \times J_2$ and $J_{\mathbb{Z}} = J_{1\mathbb{Z}} \times J_{2\mathbb{Z}}$. Here the action is diagonal on $Y_1 \times Y_2$ with Y_1 as in (1) with $J = J_1$ and Y_2 as in (2) with $J = J_2$.*

From our definition of the ordering on virtual arithmetic quotients it is clear that $A(M) \succ Y$, where Y comes from the above theorem and a linear representation of Λ. So for engaging actions on manifolds, as long as $\pi_1(M)$ has an infinite image linear representation that extends to Λ, we know $A(M)$ is nontrivial.

The proof of Theorem 6.2 also relies on the generalizations of Ratner's theorem by Shah and Witte discussed above [S,W]. These generalizations combined with results of Lubotzky and Zimmer on connected group actions [LZ1] and a detailed study of the topological structure of induced actions give the desired result. For further details the reader is referred to the article [F].

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